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Report TW 75

Solutions of the equation of Helmholtz in an angle VI

The case of a semi-infinite barrier

by

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1. Introduction

In this paper the technique which has been developed in the preceding papers of this series will be employed to solve a generalization of the well-known Sommerfeld problem of diffraction by a half-plane. The following problem will be considered

$$(1.1) \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-1\right) G(x,y,x_0,y_0) = -2\pi\delta(x-x_0)\delta(y-y_0) \; ,$$

$$(1.2) \qquad \text{for } \; y=\pm 0 \;\text{ and }\; x<0 \qquad \cos\gamma\frac{\partial G}{\partial y}-\sin\gamma\frac{\partial G}{\partial x}=0 \; ,$$

with $y_0 > 0$ and

$$(1.3) \qquad -\tfrac{1}{2}\pi < \operatorname{Re}\gamma \leq \tfrac{1}{2}\pi \; .$$

The boundary condition (1.2) applies at either side of the semi-infinite barrier along the negative X-axis.
We note that for $\gamma=0$ (a reflecting barrier) and $\gamma=\frac{1}{2}\pi$ (an absorbing barrier) the problem is equivalent to the well-known problem of diffraction of waves from a finite source by a semi-infinite screen. After the determination of the solution of the general problem in section 3 the specialization $\gamma=0$ is considered in section 4.
In section 5 we consider the behaviour of G at the upper side of the barrier. Relatively simple expressions can be derived for the value of G at the edge (0,0) and far away from the edge ($x \to -\infty$). The last section contains a few properties of some auxiliary functions which are useful in the treatment of the problem.

2. A simpler problem

We shall first solve the simpler problem of finding a solution of (1.1) in the upper half-plane $y>0$ with the boundary condition

$$(2.1) \qquad \cos\gamma\frac{\partial G}{\partial y}-\sin\gamma\frac{\partial G}{\partial x}=0 \quad \text{for } y=0.$$

We note that a particular solution of (1.1) is given by

(2.2) $$K_o(R) \text{ with } R^2 = (x-x_o)^2+(y-y_o)^2 .$$

In view of the following representation expressing the expansion of $K_o(R)$ in plane waves

(2.3) $$K_o(R) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp\left\{ -i(x-x_o)\text{shw} - |y-y_o|\,\text{chw} \right\} dw ,$$

we try the following tentative solution

(2.4) $$G=K_o(R)+ \tfrac{1}{2} \int_{-\infty}^{\infty} \exp\left\{ -i(x-x_o)\text{shw}-(y+y_o)\text{chw} \right\} g(w)dw.$$

The boundary condition (2.1) gives

$$\int_{-\infty}^{\infty} e^{-i(x-x_o)\text{shw}-y_o\,\text{chw}} \left\{ \text{ch}(w+i\gamma)-g(w)\text{ch}(w-i\gamma)\right\} dw=0$$

for all x. Therefore we have at once

(2.5) $$g(w)=\text{ch}(w+i\gamma) \,/\, \text{ch}(w-i\gamma).$$

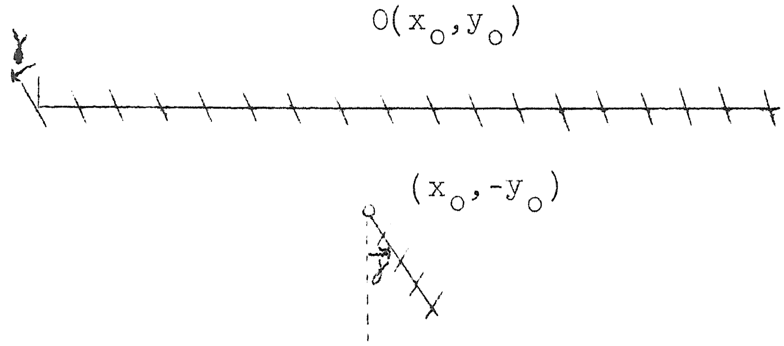
Hence the required solution, which will be denoted by G_1, is given by

(2.6)
$$G_1(x,y,x_o,y_o,\gamma)=K_o\left(\sqrt{(x-x_o)^2+(y-y_o)^2}\right) + \\ +\tfrac{1}{2} \int_{-\infty}^{\infty} \exp\left\{ -i(x-x_o)\text{shw}-(y+y_o)\text{chw}\right\} \frac{\text{ch}(w+i\gamma)}{\text{ch}(w-i\gamma)}\, dw.$$

This may also be written as

(2.7)
$$G_1(x,y,x_o,y_o,\gamma)=K_o\left(\sqrt{(x-x_o)^2+(y-y_o)^2}\right)-\left(\cos\gamma\frac{\partial}{\partial y_o} -\sin\gamma \frac{\partial}{\partial x_o}\right). \\ .\int_{0}^{\infty} K_o\left(\sqrt{(x-x_o-t \sin\gamma)^2+(y+y_o+t \cos\gamma)^2}\right)dt.$$

The latter expression may be interpreted as follows. The first term represents the influence of the pole at (x_o,y_o). The second term represents the total influence of a line of dipoles (see figure 1)



We note the following symmetry relation

(2.8) $$G_1(x,y,x_o,y_o,\gamma)=G_1(x_o,y_o,x,y,-\gamma) .$$

At y=0 we have in particular

(2.9) $$G_1(x,0,x_o,y_o,\gamma)=\cos\gamma \int_{-\infty}^{\infty} e^{-ix\,\mathrm{sh}w} \exp(ix_o\mathrm{sh}w-y_o\mathrm{ch}w)\cdot \frac{\mathrm{ch}w}{\mathrm{ch}(w-i\gamma)}\,dw.$$

It may be of interest to derive an asymptotic expression of $G_1(x,0,x_o,y_o,\gamma)$ for either $x\to+\infty$ or $x\to-\infty$. Without loss of generality we may suppose that $0<\mathrm{Re}\,\gamma\leq\frac{1}{2}\pi$. Then the asymptotic behaviour of the Fourier-integral of (2.9) is determined by the saddle points $w=\pm\frac{1}{2}i\pi$ and the pole of the integrand at $w=-i(\frac{1}{2}\pi-\gamma)$. For $x\to+\infty$ the asymptotic behaviour is determined by the pole. A simple calculation shows that for $x\to+\infty$

(2.10) $$G_1(x,0,x_o,y_o,\gamma)=\pi\sin 2\gamma\, e^{-x\cos\gamma}\exp(x_o\cos\gamma-y_o\sin\gamma)+ O(e^{-x}).$$

For $x\to-\infty$ the asymptotic behaviour is determined by the saddle point $\frac{1}{2}i\pi$. Writing

$$G_1(x,0,x_o,y_o,\gamma)=\cos\gamma\int_{-\infty}^{\infty} e^{-r\,\mathrm{ch}\,u}\exp(-x_o\mathrm{ch}u-iy_o\mathrm{sh}u)\frac{\mathrm{sh}u}{\mathrm{sh}(u-i\gamma)}du,$$

where $r=|x|$, and noting that for $r\to+\infty$

$$\int_{-\infty}^{\infty} e^{-r\,ch\,u}\, sh\,u\, f(u)du = \frac{1}{r} \int_{-\infty}^{\infty} e^{-r\,ch\,u} f'(u)\, du$$

$$\approx \frac{2}{r} f'(0)\, K_0(r) \approx \frac{1}{r} \sqrt{\frac{2\pi}{r}}\; e^{-r} f'(0) \quad ,$$

we obtain for $x \to -\infty$

$$(2.11) \quad G_1(x,0,x_0,y_0,\gamma) \approx \cot g\,\gamma(\cot g\,\gamma + y_0)\, \frac{1}{r} \sqrt{\frac{2\pi}{r}}\; e^{-r-x_0} \quad .$$

3. Solution of the problem

The solution of the problem (1.1) and (1.2) will be sought in the following form (cf also III 3.1)

$$(3.1) \qquad G = K_0(R) + \tfrac{1}{2} \int_{-\infty}^{\infty} \exp(-ix\,sh\,w - y\,ch\,w)\, g_1(w)dw$$

for $y \geq 0$, and

$$(3.2) \qquad G = K_0(R) + \tfrac{1}{2} \int_{-\infty}^{\infty} \exp(-ix\,sh\,w + y\,ch\,w)\, g_2(w)dw$$

for $y \leq 0$.

It will appear later on that the two expressions (3.1) and (3.2) can be combined into a single expression from which they can be obtained by analytic continuation.

The continuity of $\cos\gamma\, G_y - \sin\gamma\, G_x$ at the X-axis gives

$$\int_{-\infty}^{\infty} e^{-ix\,sh\,w} \left\{ ch(w-i\gamma)g_1(w) + ch(w+i\gamma)g_2(w) \right\} dw = 0$$

for all x. Therefore we have identically

$$(3.3) \qquad ch(w-i\gamma)\, g_1(w) + ch(w+i\gamma)g_2(w) = 0 \quad .$$

Therefore we may put

$$(3.4) \qquad \begin{cases} g_1(w) = ch(w+i\gamma)\; g(w) \\ g_2(w) = -ch(w-i\gamma)\; g(w) \end{cases} \quad .$$

The continuity of G at the positive X-axis gives

$$\int_{-\infty}^{\infty} e^{-ix\,sh\,w}\, g(w)ch\,w\; dw = 0 \qquad \text{for } x > 0 \;.$$

Therefore we may put

$$g(w) = \emptyset^{-}(w) \quad , \tag{3.5}$$

where $\emptyset^{-}(w)$ denotes a lower holomorphic function, i.e. holomorphic in the lower strip $-\pi < \mathrm{Im}\, w < 0$ and symmetric with respect to $-\frac{1}{2}\pi i$.

The boundary condition at $y=+0$, $x < 0$ gives when using (2.3)

$$\int_{-\infty}^{\infty} e^{-ixshw} \left\{ ch(w+i\gamma)\exp(ix_0 shw-y_0 chw)-ch(w-i\gamma)g_1(w)\right\} dw=0$$

for $x < 0$. Therefore we may put

$$ch(w+i\gamma)\exp(ix_0 shw-y_0 chw)-ch(w-i\gamma)ch(w+i\gamma)g(w)=chw\ \emptyset^{+}(w), \tag{3.6}$$

where $\emptyset^{+}(w)$ denotes an upper holomorphic function, i.e. holomorphic in the upper strip $0 < \mathrm{Im}\, w < \pi$ and symmetric with respect to $\frac{1}{2}\pi i$.

For $\mathrm{Im}\ w \to 0$ the relations (3.5) and (3.6) can be combined and we obtain the following Hilbert problem

$$chu\ \emptyset^{+}(u)+ch(u-i\gamma)ch(u+i\gamma)\emptyset^{-}(u)=ch(u+i\gamma)\exp(ix_0 shu-y_0 chu). \tag{3.7}$$

At this stage the solution of our problem has been reduced to a Hilbert problem. The latter problem, however, can be easily solved by a standard technique which involves a factorization of the Wiener-Hopf kind.

It should be noted that by the latter problem the functions $\emptyset^{+}(w)$ and $\emptyset^{-}(w)$ are not uniquely determined since any solution of (3.7) is the sum of some particular solution and a solution of the homogeneous Hilbert relation. This corresponds with the fact that the solution of (1.1) and (1.2) is the sum of some Green's function and a solution of the corresponding homogeneous problem. If we select the particular solution of (3.7) with the best possible behaviour at finity we shall obtain the Green's function with the best possible behaviour at the origin. We shall restrict our discussion here to the latter Green's function.

As is well-known the solution of (3.7) depends essentially on the factorization of $ch(u-i\gamma)ch(u+i\gamma)/chu$. The following relations show that this is elementary

$$(3.8)\begin{cases} chw=2ch\frac{1}{2}(w-\frac{1}{2}i\pi)ch\frac{1}{2}(w+\frac{1}{2}i\pi) \\ ch(w-i\gamma)ch(w+i\gamma)=sh^2w+cos^2\gamma =(shw+icos\gamma)(shw-icos\gamma). \end{cases}$$

Hence the Hilbert problem (3.7) can be put in the following simpler form

$$(3.9)\qquad \psi^+(u) + \psi^-(u) = h(u) \quad ,$$

where

$$(3.10)\begin{cases} \psi^+(w)= \dfrac{2ch\frac{1}{2}(w-\frac{1}{2}i\pi)}{shw + icos\gamma}\ \phi^+(w) \\ \psi^-(w)= \dfrac{shw - icos\gamma}{ch\frac{1}{2}(w+\frac{1}{2}i\pi)}\ \phi^-(w) \\ h(w)= \dfrac{ch(w+i\gamma)}{ch\frac{1}{2}(w+\frac{1}{2}i\pi)(shw+icos\gamma)}\ exp(ix_0shw-y_0chw) \ . \end{cases}$$

The solution of (3.9) with the best possible behaviour at infinity is

$$(3.11)\begin{cases} \psi^+(w)= \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{h(t)cht}{sht-shw}\, dt \quad , \ 0 < Imw < \pi \ , \\ \psi^-(w)= - \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{h(t)cht}{sht-shw}\, dt \ , \ -\pi < Imw < 0 \ . \end{cases}$$

The analytic continuation of the latter solutions is easy matter since the integrands are relatively simple meromorphic functions. The analytic continuation can be effected by shifting the line of integration, upwards or downwards.

The required solution of our problem can now be obtained by piecing together (2.1),(2.2),(3,4),(3.5),(3.10) and (3.11). Some simplification is obtained by using polar coordinates $r, \varphi, r_0, \varphi_0$ determined by

$$(3.12)\qquad x=r\ cos\ \varphi\ \ , y=r\ sin\ \varphi_0,\ x_0=r_0\ cos\ \varphi_0, y_0=r_0\ sin\ \varphi_0,$$

where $-\pi \leq \varphi \leq \pi$ and $0 \leq \varphi_0 \leq \pi$.

Then the expression (3.1) can be rewritten as

$$(3.13)\qquad G=K_o(R)+\tfrac{1}{2}\int_L e^{-irsh(w-i\varphi)}ch(w+i\gamma)\emptyset^-(w)dw \quad ,$$

where L is the horizontal path determined by $-\infty < \mathrm{Re}\, w < \infty$, $\mathrm{Im}\, w = c$ (= constant). The integral converges for $c < \varphi < c+\pi$. Since $ch(w+i\gamma)\emptyset^-(w)$ has no singularities for $-\pi < \mathrm{Im}\, w < 0$ the constant c may be any value in the interval $(-\pi, 0)$. Hence the integral expression holds for all values of φ, i.e. the expressions (3.13) holds in the full x,y-plane.
An equivalent expression is obtained when one starts from (3.2). The transition from either form to the other is effected by the transformation $w \rightarrow -\pi i - w$.
Another equivalent form is obtained when the integral on the right-hand side of (3.13) is replaced by one containing the function $\emptyset^+(w)$. It follows from (3.7) and (3.13) that

$$(3.14)\qquad G=G_1-\tfrac{1}{2}\int_L e^{-irsh(w-i\varphi)}\,\frac{chw}{ch(w-i\gamma)}\,\emptyset^+(w)dw \quad ,$$

where G_1 is given by (2.6) .
The latter expression might be useful at the upper side of the negative X-axis. Then the integral may be interpreted as the disturbance from the edge at (0,0).

The expressions (3.13) and (3.14) will be written in the following more symmetrical way which is obtained by making the transformations $w \rightarrow w-\tfrac{1}{2}i\pi$ and $t \rightarrow t+\tfrac{1}{2}i\pi$.

$$(3.15)\qquad G=K_o(R)+\frac{1}{2\pi}\int_L\int_{L_o} \exp\left\{-rch(w-i\varphi)-r_o ch(w_o+i\varphi_o)\right\}.$$

$$.\, Q(w,w_o)dw\, dw_o \quad ,$$

where

$$(3.16)\quad Q(w,w_o)=\frac{sh(w+i\gamma)sh(w_o+i\gamma)ch\tfrac{1}{2}w\; ch\tfrac{1}{2}w_o}{(chw+\cos\gamma)(chw_o+\cos\gamma)(chw+chw_o)} \quad ,$$

and where the horizontal paths L and L_o are respectively $\mathrm{Im}\, w=\tfrac{1}{2}\pi-\varepsilon$, and $\mathrm{Im}\, w_o= -\tfrac{1}{2}\pi+\varepsilon$ with ε a sufficiently small

positive number.
Of course one may shift L and L_o upwards and downwards as long as no poles are passed. This makes it possible to extend the expression for all values of φ and φ_o.

The expression (3.14) becomes similarly

$$\text{(3.17)}\quad G=G_1+\frac{1}{2\pi}\int_L\int_{L_o}\exp\left\{-r\,ch(w-i\varphi)-r_o ch(w_o+i\varphi_o)\right\}Q(w,w_o)dwdw_o,$$

where now

$$L(\mathrm{Im}w=\tfrac{1}{2}\pi+\varepsilon)\quad,\quad L_o(\mathrm{Im}w_o=-\tfrac{1}{2}\pi-\varepsilon)\;.$$

We note that the expressions (3.16) and (3.17) differ only in the explicit contribution of the pole $w=i\pi+w_o$ due to the factor $chw+chw_o$ in the denominator of Q .

It follows from (3.15) that as in (2.8) the following symmetry relation holds

$$\text{(3.18)}\qquad G(x,y,x_o,y_o,\gamma)=G(x_o,y_o,x,y,-\gamma).$$

4. The special case $\gamma=0$

This is the well-known problem of diffraction of e.g. a sound pulse around a semi-infinite screen. The solution of this problem was first obtained by Sommerfeld (1901) and Macdonald (1902).

Here we may use the solution in the form (3.15). Substitution of $\gamma=0$ gives

$$Q(w,w_o)=\frac{sh\frac{1}{2}w\;sh\frac{1}{2}w_o}{chw+chw_o}=\frac{1/2}{ch\frac{1}{2}(w-w_o)}-\frac{1/2}{ch\frac{1}{2}(w+w_o)}\;.$$

Introducing the auxiliary function

$$\chi(r,r_o,\alpha)=\frac{1}{4\pi}\iint_{-\infty}^{\infty}e^{-rchu-r_ochu_o}\frac{1}{ch\frac{1}{2}(u+u_o+i\alpha)}du\,du_o=$$

$$\text{(4.1)}\qquad =\int_{r+r_o}^{\infty}\frac{e^{-t}dt}{\sqrt{t^2-r^2-r_o^2+2rr_o\cos\alpha}}\;,$$

which is treated more fully in the Appendix, the solution is obtained in the form

(4.2) $G(x,y,x_o,y_o,0)=K_o(R)+\frac{1}{2}\chi(r,r_o,\varphi+\varphi_o)-\frac{1}{2}\chi(r,r_o,\varphi-\varphi_o)$.

We note in passing the somewhat surprising fact that

(4.3) $\frac{1}{2}\{G(x_1+0,x_o,y_o,0)+G(x_1-0,x_o,y_o,0)\}=K_o(\sqrt{(x-x_o)^2+y_o^2})$.

It follows from (4.2) or from (4.3) that in particular

(4.4) $G(0,0,x_o,y_o-0)=K_o(r_o)$,

which was to be expected by virtue of the symmetry of the Green's function.

According to the relation (6.10) of the appendix we find in particular at the upper side of the barrier

$$G(r,\pi,r_o,\varphi_o,0)=\chi(r,r_o,\varphi_o+\pi)=$$
$$=2\,K_o(\sqrt{r^2+r_o^2+2r\,r_o\cos\varphi_o})-\chi(r,r_o,\pi-\varphi_o)\quad .$$

It follows from (6.13) of the appendix that for $r\to\infty$ we have the following asymptotic relation

(4.6) $G(r,\pi,r_o,\varphi_o,0)\approx\sqrt{\frac{\pi}{2r}}\,e^{-r}\left\{2e^{-x_o}-\psi(r_o,i\pi-i\varphi_o)\right\}$.

Substitution of the formula (6.1) defining ψ gives

(4.7) $G(r,\pi,r_o,\varphi_o,0)\approx\sqrt{\frac{\pi}{2r}}\;e^{-r}\;(1+\mathrm{erf}\sqrt{r_o-x_o})$.

5. Behaviour at the upper side of the barrier

We shall first consider the value of G at the origin. From (3.15) it follows that

(5.1) $G(0,0,x_o,y_o,\gamma)=K_o(r_o)+\frac{1}{2\pi}\int_{L_o}e^{-r_o\mathrm{ch}(w_o+i\varphi_o)}k(w_o)dw_o$,

where

$$(5.2)\quad k(w_o)=\frac{sh(w_o+i\gamma)ch\frac{1}{2}w_o}{chw_o+\cos\gamma}\int_L \frac{sh(w+i\gamma)ch\frac{1}{2}w}{(chw+\cos\gamma)(chw+chw_o)}\,dw =$$

$$=\frac{i\,\sin\gamma\,ch\frac{1}{2}w_o}{sh(w_o-i\gamma)}\int_L\left\{\frac{chw_o}{chw+chw_o}-\frac{\cos\gamma}{chw+\cos\gamma}\right\}ch\tfrac{1}{2}w\,dw =$$

$$=\frac{i\pi\,\sin\gamma\,ch\frac{1}{2}w_o}{sh(w_o-i\gamma)}\left\{\frac{chw_o}{ch\frac{1}{2}w_o}-\frac{\cos\gamma}{\cos\frac{1}{2}\gamma}\right\} =$$

$$=\pi\left\{-\tfrac{1}{2}+\tfrac{1}{2}\frac{sh(w_o+i\gamma)}{sh(w_o-i\gamma)}\right. +\pi\sin\tfrac{1}{2}\gamma\cos\gamma\left\{\frac{\sin\frac{1}{2}\gamma}{ch\frac{1}{2}(w_o-i\gamma)}+\right.$$

$$\left.+\frac{\cos\frac{1}{2}\gamma}{ch\frac{1}{2}(w_o+i\pi-i\gamma)}\right\} .$$

Then by using (2.6) and (6.6) we obtain

$$(5.3)\quad G(0,0,x_o,y_o,\gamma)=\tfrac{1}{2}G_1(0,0,x_o,y_o,\gamma)+$$

$$+\pi\sin\tfrac{1}{2}\gamma\cos\gamma\left\{\sin\tfrac{1}{2}\gamma\,\psi(r_o,i\varphi_o+i\gamma)+\right.$$

$$\left.+\cos\tfrac{1}{2}\gamma\,\psi(r_o,i\pi-i\varphi_o-i\gamma)\right\} .$$

Next we consider the behaviour at $\varphi=\pi$ for $r\to\infty$. We obtain from (3.14)

$$(5.4)\quad G(r,\pi,r_o,\varphi_o,\gamma)=$$

$$=G_1(r,\pi,r_o,\varphi_o,\gamma)-\tfrac{1}{2}\int_{-\infty}^{\infty}e^{irshw}\,\frac{chw}{ch(w-i\gamma)}\,\phi^+(w)dw.$$

If $0<\mathrm{Re}\,\gamma\leq\frac{1}{2}\pi$ the asymptotic expansion of the second term on the right-hand side is determined by the saddle point at $w=\frac{1}{2}\pi i$. This gives, in view of (2.11)

$$(5.5)\quad G(r,\pi,r_o,\varphi_o,\gamma)\approx \cot g\,\gamma\,\frac{1}{r}\sqrt{\frac{2\pi}{r}}\,e^{-r}\,\cdot$$

$$\cdot\left\{(\cot g\,\gamma+y_o)e^{-x_o}-\frac{1}{2\sin\gamma}\,\phi^+(\tfrac{1}{2}\pi i)\right\} .$$

If $-\frac{1}{2}\pi<\mathrm{Re}\,\gamma<0$ the asymptotic behaviour is determined by the

pole of the integrand at $w=i(\frac{1}{2}\pi+\gamma)$.
Then we have in view of (2.10)

$$(5.6)\quad G(r,\pi,r_0,\varphi_0,\gamma)\sim \left\{-\pi\sin 2\gamma\; e^{-x_0\cos\gamma+y_0\sin\gamma}+\pi\sin\gamma\,\phi^+(\tfrac{1}{2}i\pi+i\gamma)\right\}\cdot e^{-r\cos\gamma}+O(e^{-r}).$$

6. Auxiliary functions

The auxiliary function $\psi(r,z)$ is defined for $r\geq 0$ and all complex values of z by

$$(6.1)\qquad \psi(r,z)=e^{r\,\mathrm{ch}z}\,\mathrm{erfc}(\sqrt{2r}\,\mathrm{ch}\tfrac{1}{2}z).$$

In order to facilitate reading we mention a few well-known properties of the error functions.

$$(6.2)\qquad \mathrm{erf}\,z=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt\ ,$$

$$(6.3)\qquad \mathrm{erfc}\,z=1-\mathrm{erf}\,z\ ,$$

$$(6.4)\qquad \mathrm{erfc}\,z+\mathrm{erfc}(-z)=2$$

$$(6.5)\quad \mathrm{erfc}\,z\sim\frac{e^{-z^2}}{z\sqrt{\pi}}\quad \text{for } |z|\to\infty \text{ in the sector } |\arg z|<\tfrac{3}{4}\pi\ .$$

The function $\psi(r,z)$ may be represented by the following integral expression

$$(6.6)\qquad \psi(r,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-r\,\mathrm{ch}t}\,\frac{1}{\mathrm{ch}\frac{1}{2}(t+z)}\,dt\ ,$$

where z is restricted to the strip $|\mathrm{Im}\,z|<\pi$.
A proof of (6.6) is easily obtained by verifying that

$$\frac{d}{dr}\left(e^{-r\,\mathrm{ch}z}\,\psi(r,z)\right)=-\frac{2}{\sqrt{\pi}}\;e^{-r(1+\mathrm{ch}z)}\ .$$

The relation (6.4) gives

$$(6.7)\qquad \psi(r,z-i\pi)+\psi(r,z+i\pi)=2\,e^{-r\,\mathrm{ch}z}\ .$$

Since $\psi(r,z)$ is even in z and periodic with the period $4\pi i$ the function is completely determined by (6.6) and (6.7). The asymptotic relation (6.5) gives

$$(6.8) \qquad \psi(r,z) \approx \frac{1}{\operatorname{ch}\frac{1}{2}z} \frac{e^{-r}}{\sqrt{2\pi r}} \quad ,$$

which is valid for Re $z \to \pm \infty$, $|\operatorname{Im} z| < \frac{3}{2}\pi$.

The auxiliary function $\chi(r,r_o,\alpha)$ is defined for $r \geq 0$ and $r_o \geq 0$ by

$$(6.9) \quad \chi(r,r_o,\alpha)=\tfrac{1}{2} \int_{-\infty}^{\infty} e^{-r\operatorname{ch}t} \psi(r_o,t+i\alpha)dt \ .$$

It follows from the properties of ψ that χ is an even function of α and periodic with the period 4π . The relation (6.7) gives

$$(6.10) \quad \chi(r,r_o,\alpha+2\pi)+\chi(r,r_o,\alpha)=2K_o(\sqrt{r^2+r_o^2-2r\,r_o\cos\alpha}) \ .$$

Substitution of the expression (6.6) in (6.9) gives

$$(6.11) \qquad \chi(r,r_o,\alpha)=\frac{1}{4\pi} \iint_{-\infty}^{\infty} e^{-r\operatorname{ch}t-r_o\operatorname{ch}t_o} \frac{1}{\operatorname{ch}\frac{1}{2}(t+t_o+i\alpha)}\,dtdt_o$$

where $|\operatorname{Re}\alpha|<\pi$. The latter formula shows that χ is symmetric in r and r_o.

We note the following particular case which easily follows from (6.9)

$$(6.12) \qquad \chi(r,0,\alpha)=K_o(r) \ .$$

We have further the following extremely useful representation of χ as a single integral due originally to Macdonald

$$(6.12) \qquad \chi(r,r_o,\alpha)=\int_{r+r_o}^{\infty} \frac{e^{-t}}{\sqrt{t^2-R^2}}\,dt \ ,$$

where $R^2=r^2+r_o^2-2r\,r_o\cos\alpha$.

A simple proof of (6.12) is obtained as follows. Consider the triangle A B B_o formed by the sides AB=r, $AB_o=r_o$ and $\angle BAB_o=\alpha$. We note that $BB_o=R$.

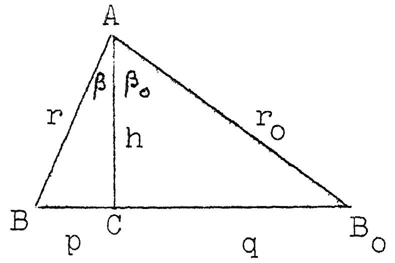


fig. 2

If C is the projection of A upon BB_o we introduce the quantities β, β_o, h, p and q as shown in figure 2. The function χ of the three variables r, r_o and α may be considered as a function of the three other independent variables p, q and h. A simple calculation shows that

$$\frac{\partial r}{\partial h} = \cos\beta \quad , \quad \frac{\partial r_o}{\partial h} = \cos\beta_o \quad , \quad \frac{\partial\alpha}{\partial h} = -\frac{\sin\beta}{r} - \frac{\sin\beta_o}{r_o} \; .$$

Then we obtain from (6.11)

$$\frac{\partial\chi}{\partial h} = \cos\beta\,\frac{\partial\chi}{\partial r} + \cos\beta_o\,\frac{\partial\chi}{\partial r_o} - \left(\frac{\sin\beta}{r}\,\frac{\partial\chi}{\partial\alpha} + \frac{\sin\beta_o}{r_o}\,\frac{\partial\chi}{\partial\alpha}\right) =$$

$$= -\frac{1}{4\pi}\iint_{-\infty}^{\infty} e^{-r\,\mathrm{ch}t - r_o\,\mathrm{ch}t_o}\,\frac{\mathrm{ch}(t+i\beta)+\mathrm{ch}(t_o+i\beta_o)}{\mathrm{ch}\frac{1}{2}(t+t_o+i\alpha)}\,dt\,dt_o =$$

$$= -\frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{-r\,\mathrm{ch}t - r_o\,\mathrm{ch}t_o}\,\mathrm{ch}\tfrac{1}{2}(t-t_o+i\beta-i\beta_o)\,dt\,dt_o =$$

$$= -\frac{\cos\frac{1}{2}(\beta-\beta_o)}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-r\,\mathrm{ch}t - r_o\,\mathrm{ch}t_o}\,\mathrm{ch}\tfrac{1}{2}t\;\mathrm{ch}\tfrac{1}{2}t_o\;dt\,dt_o =$$

$$= -\frac{\cos\frac{1}{2}(\beta-\beta_o)}{\sqrt{rr_o}}\,e^{-r-r_o} = -\frac{\cos\beta+\cos\beta_o}{\sqrt{(r+r_o)^2-R^2}}\,e^{-r-r_o} =$$

$$= -\frac{e^{-(r+r_o)}}{\sqrt{(r+r_o)^2-R^2}}\,\frac{\partial(r+r_o)}{h} \quad .$$

The result (6.12) now follows at once by integration. The asymptotic behaviour of χ follows easily from (6.8). In fact we have

$$(6.13) \quad \chi(r,r_o,\alpha) \approx \sqrt{\frac{\pi}{2r_o}}\, e^{-r_o}\, \psi(r,i\alpha)$$

for $r_o \to \infty$ and $|\mathrm{Re}\,\alpha| < \pi$.

References

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